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On the Unique Solvability of a Nonlinear Functional Evolution Equation

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Existence and uniqueness of weak solutions to an abstract functional evolution equation are proved. An application of this theory to a nonlinear nonlocal reaction-diffusion equation is presented.

Key Words: functional evolution equations; existence-uniqueness; nonlocal reaction-diffusion equations.

1. INTRODUCTION

In this paper, we study the existence and uniqueness of solutions to the abstract functional differential equation

$$\begin{gathered} \dot{u}(t) = g(l(u(t)))A(t)u(t) + F(t, u(t)), \qquad t \in (0, T), \\ u(0) = u_0. \end{gathered} \tag{1.1}$$

Here l is a continuous linear functional on a separable Hilbert space, and A is a time-dependent linear operator. The nonlinear functions g and F are assumed to satisfy some Lipschitz continuity conditions (see Section 2 for the precise assumptions).

In order to motivate the main results for Problem (1.1), we recall some recent results for two related problems. On the one hand, Chipot and Lovat in [2] studied the diffusion problem

$$\begin{array}{ll} u_t = g(l(u(t,\cdot)))\Delta u + h(t), & t \in (0,T),\ x \in \Omega \\ u(t,x) = 0, & t \in (0,T),\ x \in \partial\Omega, \\ u(0) = u_0(x) & x \in \Omega. \end{array} \tag{1.2}$$

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Here, Ω is a smooth bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, and l is a continuous linear functional on $L^2(\Omega)$. The function g satisfies similar conditions as stated in Section 2 of this paper. The authors of [2] proved the existence and uniqueness of solutions to (1.2) using a Galerkin approximation argument. In particular, they established the weak convergence of the Galerkin approximation and showed that the limit is a solution. They also established some asymptotic results for special cases of g. On the other hand, Ackleh and Ke [1] studied the following autonomous abstract equation on a Banach space B:

$$\begin{aligned} \dot{u}(t) &= \frac{1}{a(u(t))} Au(t) + F(u(t)), \qquad t \in (0, T), \\ u(0) &= u_0. \end{aligned} \tag{1.3}$$

Here the linear time-independent operator A generates a strongly continuous semigroup of linear operators on B. The functional a: $B \to [0, \infty)$ is locally Lipschitz continuous satisfying $a(\xi) > 0$, for $\xi \neq 0$ and $a(0) \geq 0$. The time-independent function F is locally Lipschitz continuous. The authors in [1] transformed (1.3) into a nonlocal semilinear problem and used the semigroup of linear operators theory (e.g. see [4, 5]) to show the local (in time) existence and uniqueness of solutions to the transformed problem. Then they argued the local existence and uniqueness of solutions to (1.3).

The goal of this paper is to extend such existence–uniqueness results to the nonautonomous problem (1.1). Our approach is in the spirit of [2], which is based on a Galerkin approximation method. Unlike the problem (1.2) studied in [2], for the problem (1.1) weak convergence of the Galerkin approximation is not enough to pass to the limit in the nonlinear function F. To achieve this we will prove that our sequence of Galerkin approximation converges strongly. We remark that in [3, p. 520] such a result was established for the linear case of (1.1) where $g(l(u(t))) = 1$ and $F(t, u(t)) = F(t)$.

2. EXISTENCE–UNIQUENESS

Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|_H$. Let V be a separable Hilbert space which is densely and continuously imbedded in H, with norm $\|\cdot\|_V$ and imbedding constant k; that is, for each $\phi \in V$, we have $\|\phi\|_H \leq k\|\phi\|_V$. We use these spaces to form a Gelfand triple structure $V \hookrightarrow H = H^* \hookrightarrow V^*$. We assume that l is a continuous linear functional of H. The operator $A(t)$ is defined (under the assumptions below) in terms of a time-dependent sesquilinear form σ: $[0, T] \times V \times V \to \mathbb{R}$; that is, for each $t \in [0, T]$, $A(t) \in \mathscr{L}(V, V^*)$ and $\langle -A(t)\phi, \psi \rangle_{V^*, V} = \sigma(t)(\phi, \psi)$. Assume that the sesquilinear form σ and the nonlinear functions F and g satisfy the following conditions:

(A1) The function $\sigma(\cdot)(\phi, \psi)$ is measurable on $[0, T]$, for fixed $\phi, \psi \in V$.

(A2) There exists a positive constant k_0 such that $|\sigma(t)(\phi, \psi)| \leq k_0 \|\phi\|_V \|\psi\|_V \ \forall \phi, \psi \in V$ uniformly in t on $[0, T]$.

(A3) There exists a positive constant c_0 such that $\sigma(t)(\phi, \phi) \geq c_0 \|\phi\|_V^2, \ \forall \phi \in V$ uniformly in t on $[0, T]$.

(A4) The function F: $[0, T] \times H \to H$ is continuous. Moreover, F is locally Lipschitz continuous in H uniformly in t on $[0, T]$. Furthermore, for any $t \in [0, T]$ we have $\|F(t, y)\|_H \leq a_1 \|y\|_H + a_2$, for some $a_1, a_2 > 0$.

(A5) The function g: $\mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. Furthermore, there exist positive numbers m and M such that $m \leq g(\xi) \leq M$, for all $\xi \in \mathbb{R}$.

As in the linear case (see, e.g., [3, 6]), we seek solutions of (1.1) in the Hilbert space

$$W(0, T) = \{u : u \in L^2(0, T; V), \dot{u} \in L^2(0, T; V^*)\},$$

whose norm is given by

$$\|u\|_W^2 = \int_0^T \|u(t)\|_V^2 + \|\dot{u}(t)\|_{V^*}^2 \, dt.$$

Recall that $W(0, T)$ is continuously imbedded in the space $C([0, T]; H)$. To this end, we denote by $\mathscr{D}'(0, T)$ the space of distributions on $(0, T)$ and we define a weak solution of the problem (1.1).

DEFINITION 2.1. We say that a function $u \in W(0, T)$ is a weak solution of (1.1) if it satisfies

$$\begin{aligned} &\langle \dot{u}(\cdot), \phi \rangle_{V^*, V} + g(l(u(\cdot)))\sigma(\cdot)(u(\cdot), \phi) \\ &\quad = \langle F(\cdot, u(\cdot)), \phi \rangle \text{ in the sense of } \mathscr{D}'(0, T) \text{ for all } \phi \in V, \qquad (2.1) \\ &u(0) = u_0. \end{aligned}$$

Next we will prove the existence and uniqueness of weak solutions to (1.1).

THEOREM 2.2. *Suppose that the hypotheses* (A1)–(A5) *hold. Then the problem* (1.1) *has a unique weak solution.*

Proof. We consider the following Galerkin approximation: Let $\{\psi^k\}_{k=1}^\infty \subset V$ be a total linearly independent system in V, for example, an orthonormal basis. We define an approximating solution to the problem (1.1) by

$$u_n(t) = \sum_{k=1}^n \eta_n^k(t) \psi^k,$$

where the coefficients $\{\eta_n^k(t)\}_{k=1}^n$ are chosen so that $u_n(t)$ is the unique solution of the n-dimensional system

$$\begin{aligned}\langle \dot{u}_n(t), \psi^j\rangle_{V^*,V} &+ g(l(u_n(t)))\sigma(t)(u_n(t), \psi^j)\\ &= \langle F(t, u_n(t)), \psi^j\rangle, \qquad j = 1, \ldots, n,\end{aligned} \tag{2.2}$$

satisfying the initial condition

$$\eta_n^k(0) = \eta_n^{k,0}.$$

The set $\{\eta_n^{k,0}\}$ is chosen such that

$$u_{n,0} = \sum_{k=1}^{n} \eta_n^{k,0}\psi^k \to u_0 \text{ in } H, \qquad \text{as } n \to \infty. \tag{2.3}$$

Multiplying the equation (2.2) by $\eta_n^j(t)$ and summing over $j = 1, \ldots, n$, we obtain

$$\begin{aligned}\langle \dot{u}_n(t), u_n(t)\rangle_{V^*,V} &+ g(l(u_n(t)))\sigma(t)(u_n(t), u_n(t))\\ &= \langle F(t, u_n(t)), u_n(t)\rangle.\end{aligned} \tag{2.4}$$

Integrating from 0 to t and using (A3) we get

$$\int_0^t \frac{1}{2}\frac{d}{ds}\|u_n(s)\|_H^2\, ds + c_0 m \int_0^t \|u_n(s)\|_V^2\, ds \le \int_0^t \|F(s, u_n(s))\|_H \|u_n(s)\|_H\, ds.$$

Using the inequality $2|ab| \le a^2 + b^2$ together with (A4) we get

$$\begin{aligned}&\|u_n(t)\|_H^2 + 2mc_0 \int_0^t \|u_n(s)\|_V^2\, ds\\ &\quad\le \int_0^t (a_1\|u_n(s)\|_H + a_2)^2\, ds + \int_0^t \|u_n(s)\|_H^2\, ds + \|u_{n,0}\|_H^2.\end{aligned}$$

In view of (2.3), we see that

$$\begin{aligned}&\|u_n(t)\|_H^2 + 2mc_0 \int_0^t \|u_n(s)\|_V^2\, ds\\ &\quad\le \int_0^t (2a_1^2\|u_n(s)\|_H^2 + 2a_2^2)\, ds + \int_0^t \|u_n(s)\|_H^2\, ds + \|u_{n,0}\|_H^2\\ &\quad\le \int_0^t (2a_1^2 + 1)\|u_n(s)\|_H^2\, ds + C_1,\end{aligned}$$

for some positive constant C_1. Ignoring the second term on the left-hand side and using Gronwall's lemma we see that there exists a positive constant C_2 such that $\|u_n(t)\|_H^2 \le C_2$, for $t \in [0, T]$. From this it follows that for any $t \in [0, T]$,

$$\|u_n(t)\|_H^2 + 2mc_0 \int_0^t \|u_n(s)\|_V^2\, ds \le C_3.$$

Hence, we can claim that there exists a subsequence (denoted again by u_n) and a limit function u such that $u_n \to u$ weakly in $L^2(0, T; V)$ and weakly* in $L^\infty(0, T; H)$. From this one can verify that $\dot{u} \in L^2(0, T; V^*)$. Hence, $u \in W(0, T)$. Furthermore, we have $l(u_n) \to l(u)$ in $L^2(0, T)$. Using (A4) we see that for any $t \in [0, T]$, $\|F(t, u_n(t))\|_H \leq a_1 \|u_n(t)\|_H + a_2$. Hence, it easily follows that $F(\cdot, u_n) \to f$ weakly in $L^2(0, T; H)$. Now by a standard argument (see, e.g., [3, p. 517]) we can show that the limit satisfies the equation

$$\begin{aligned}\langle \dot{u}(\cdot), \phi \rangle_{V^*, V} &+ g(l(u(\cdot)))\sigma(\cdot)(u(\cdot), \phi) \\ &= \langle f(\cdot), \phi \rangle \text{ in the sense of } \mathscr{D}'(0, T) \text{ for all } \phi \in V, \\ & \qquad u(0) = u_0.\end{aligned} \tag{2.5}$$

To show that the limit u is a weak solution to (1.1) we need to prove that $f(t) = F(t, u(t))$ a.e. $t \in [0, T]$. To achieve this we prove that the above weak convergence is in fact a strong one. To this end, subtract (2.5) (with ϕ replaced by u) from (2.4) and integrate from 0 to t to get

$$\begin{aligned}&\frac{1}{2}\|u_n(t)\|_H^2 - \frac{1}{2}\|u(t)\|_H^2 - \frac{1}{2}\|u_{n,0}\|_H^2 + \frac{1}{2}\|u_0\|_H^2 \\ &\quad + \int_0^t g(l(u_n(s)))\sigma(s)(u_n(s), u_n(s))\, ds - \int_0^t g(l(u(s)))\sigma(s)(u(s), u(s))\, ds \\ &\quad = \int_0^t \langle F(s, u_n(s)), u_n(s) \rangle ds - \int_0^t \langle f(s), u(s) \rangle\, ds.\end{aligned}$$

By adding and subtracting a few terms we obtain

$$\begin{aligned}&\frac{1}{2}\langle u_n(t) - u(t), u_n(t) - u(t) \rangle + \langle u_n(t), u(t) \rangle - \|u(t)\|_H^2 - \frac{1}{2}\|u_{n,0}\|_H^2 \\ &\quad + \frac{1}{2}\|u_0\|_H^2 + \int_0^t g(l(u_n(s)))\sigma(s)(u_n(s) - u(s), u_n(s) - u(s))\, ds \\ &\quad + 2\int_0^t g(l(u_n(s)))\sigma(s)(u_n(s), u(s))\, ds \\ &\quad - \int_0^t g(l(u_n(s)))\sigma(s)(u(s), u(s))\, ds - \int_0^t g(l(u(s)))\sigma(s)(u(s), u(s))\, ds \\ &\quad = \int_0^t \langle F(s, u_n(s)) - F(s, u(s)), u_n(s) - u(s) \rangle\, ds \\ &\qquad + \int_0^t \langle F(s, u_n(s)), u(s) \rangle\, ds + \int_0^t \langle F(s, u(s)), u_n(s) \rangle\, ds \\ &\qquad - \int_0^t \langle F(s, u(s)), u(s) \rangle\, ds - \int_0^t \langle f(s), u(s) \rangle\, ds.\end{aligned}$$

Using the local Lipschitz property of F (recall that for any $t \in [0, T]$, $\|u_n(t)\|_H^2 \leq C_2$) and simplifying, we get that there exists a positive constant C_4 such that

$$\frac{1}{2}\|u_n(t) - u(t)\|_H^2 + mc_0 \int_0^t \|u_n(s) - u(s)\|_V^2 \, ds \leq C_4 \int_0^t \|u_n(s) - u(s)\|_H^2 \, ds + X_n(t), \tag{2.6}$$

where

$$\begin{aligned} X_n(t) = &-\langle u_n(t), u(t)\rangle + \|u(t)\|_H^2 - 2\int_0^t g(l(u_n(s)))\sigma(s)(u_n(s), u(s)) \, ds \\ &+ \int_0^t g(l(u_n(s)))\sigma(s)(u(s), u(s)) \, ds \\ &+ \int_0^t g(l(u(s)))\sigma(s)(u(s), u(s)) \, ds \\ &+ \int_0^t \langle F(s, u_n(s)), u(s)\rangle ds + \int_0^t \langle F(s, u(s)), u_n(s)\rangle ds \\ &- \int_0^t \langle F(s, u(s)), u(s)\rangle ds - \int_0^t \langle f(s), u(s)\rangle ds \\ &+ \frac{1}{2}\|u_{n,0}\|_H^2 - \frac{1}{2}\|u_0\|_H^2. \end{aligned}$$

Using (2.3) and the weak convergence established above we can verify that $X_n(t) \to 0$ as $n \to \infty$, for $t \in [0, T]$. Hence, ignoring the second term in (2.6) we can show using Gronwall's lemma that for every $t \in [0, T]$, $\|u_n(t) - u(t)\|_H \to 0$ as $n \to \infty$. From this it follows that for any $t \in [0, T]$,

$$\frac{1}{2}\|u_n(t) - u(t)\|_H^2 + mc_0 \int_0^t \|u_n(s) - u(s)\|_V^2 \, ds \to 0, \qquad \text{as } n \to \infty.$$

This proves that for all $t \in [0, T]$, $u_n(t) \to u(t)$ strongly in H. Furthermore, $u_n \to u$ strongly in $L^2(0, T; V)$. Using (A4) we can show that $F(\cdot, u_n) \to F(\cdot, u)$ strongly in $L^2(0, T; H)$. Hence, $f(t) = F(t, u(t))$ a.e. $t \in [0, T]$ and u is a weak solution of (1.1). As for uniqueness, assume that Eq. (1.1) has two weak solutions u and v. Subtract the equation (2.1) for u from that for v, let $\phi = u(t) - v(t)$, and use techniques similar to those above to obtain for each $t \in [0, T]$,

$$\begin{aligned} &\frac{1}{2}\|u(t) - v(t)\|_H^2 + mc_0 \int_0^t \|u(s) - v(s)\|_V^2 \, ds \\ &\quad \leq \int_0^t k_0 |g(l(u(s))) - g(l(v(s)))| \|v(s)\|_V \|u(s) - v(s)\|_V \, ds \\ &\qquad + C_5 \int_0^t \|u(s) - v(s)\|_H^2 \, ds, \end{aligned}$$

for some positive constant C_5. Simplifying and applying Young's inequality we obtain

$$\begin{aligned}
&\frac{1}{2}\|u(t)-v(t)\|_H^2 + mc_0\int_0^t \|u(s)-v(s)\|_V^2\,ds \\
&\quad \le C_6\int_0^t \|u(s)-v(s)\|_H\|v(s)\|_V\|u(s)-v(s)\|_V\,ds \\
&\qquad + C_5\int_0^t \|u(s)-v(s)\|_H^2\,ds \\
&\quad \le \int_0^t \left(\frac{(C_6)^2}{2c_0 m}\|v(s)\|_V^2 + C_5\right)\|u(s)-v(s)\|_H^2\,ds \\
&\qquad + \frac{c_0 m}{2}\int_0^t \|u(s)-v(s)\|_V^2\,ds.
\end{aligned}$$

From this we have

$$\begin{aligned}
&\frac{1}{2}\|u(t)-v(t)\|_H^2 + \frac{mc_0}{2}\int_0^t \|u(s)-v(s)\|_V^2\,ds \\
&\quad \le \int_0^t \mu(s)\|u(s)-v(s)\|_H^2\,ds,
\end{aligned}$$

where $\mu \in L^1(0,T)$. Once again, ignoring the second term and applying Gronwall's lemma, we have $\|u(t)-v(t)\|_H = 0$ for every $t \in [0,T]$. Hence, uniqueness is established. ∎

Remark 2.3. In our argument above we proved that, for all $t \in [0,T]$, $u_n(t) \to u(t)$ strongly in H and also that $u_n \to u$ strongly in $L^2(0,T;V)$, along a subsequence. However, due to the uniqueness of the weak solution u we can conclude that the entire sequence u_n converges strongly as well. This result is important, particularly when the Galerkin approximation method developed here is used to compute solutions of (1.1).

Next we establish a local existence result. To this end, we relax the assumptions on F and g.

($\overline{\text{A4}}$) The function F: $[0,T]\times H \to H$ is continuous. Moreover, F is locally Lipschitz continuous in H uniformly in t on $[0,T]$.

($\overline{\text{A5}}$) The function g: $[\alpha,\beta] \to (0,\infty)$ is Lipschitz continuous.

THEOREM 2.4. *Let u_0 be such that $l(u_0) \in (\alpha,\beta)$. Suppose that the hypotheses* (A1)–(A3) *and* ($\overline{\text{A4}}$)–($\overline{\text{A5}}$) *hold. Then there exists a* $0 < T^* \le T$ *such that the problem* (1.1) *has a unique weak solution on* $[0,T^*]$ (*i.e.*, *a function u which satisfies Definition* 2.1 *with T replaced with T^**).

Proof. Let P be the Hilbert space radial retraction onto the ball (in H) of radius 1 centered at u_0. Define, for each $t \in [0, T]$, $\overline{F}(t, \xi) = F(t, P\xi)$. Note that from (A4) we can show that $\overline{F}$ satisfies the global Lipschitz condition

$$\|\overline{F}(t, v) - \overline{F}(t, z)\|_H \leq L_{B_{(1+|u_0|)}} \|Pv - Pz\|_H \leq L\|v - z\|_H, \qquad \text{for all } v, z \in H.$$

Here, $L_{B_{(1+|u_0|)}}$ is the local Lipschitz constant of the function F on the ball (in H) of radius $1 + |u_0|$ centered at 0. Define the Lipschitz continuous extension $\bar{g}$ of g as

$$\bar{g}(\xi) = \begin{cases} g(\alpha) & \xi < \alpha, \\ g(\xi) & \alpha \leq \xi \leq \beta, \\ g(\beta) & \xi > \beta. \end{cases}$$

Now, consider the following abstract evolution equation:

$$\begin{aligned} \dot{u}(t) &= \bar{g}(l(u(t)))A(t)u(t) + \overline{F}(t, u(t)), \qquad t \in (0, T), \\ u(0) &= u_0. \end{aligned} \tag{2.7}$$

We can verify that $\overline{F}$ and $\bar{g}$ satisfy (A4) and (A5), respectively. Then by Theorem 2.2 the problem (2.7) has a unique solution u on $[0, T]$. Since $u \in C([0, T]; H)$ then $l(u) \in C[0, T]$. This implies that there exists a T_1^* such that $l(u(t))$ remains in (α, β) (since $l(u_0) \in (\alpha, \beta)$). Similarly, there exists a T_2^* such that $\|u(t) - u_0\|_H \leq 1$ for all $t \in [0, T_2^*]$. Let $T^* = \min\{T_1^*, T_2^*\}$. Then for all $t \in [0, T^*]$, $\overline{F}(t, u(t)) = F(t, u(t))$ and $\bar{g}(l(u(t))) = g(l(u(t)))$. This implies that u is a solution to the problem (1.1) on $[0, T^*]$. Uniqueness follows using an argument similar to that above. ∎

3. AN EXAMPLE

We consider the following nonlinear nonlocal reaction-diffusion equation which arises in population dynamics:

$$\begin{aligned} &u_t = g(l(u(t, \cdot)))\Delta u + F(t, u) && t \in (0, T),\ x \in \Omega, \\ &u(t, x) = 0 && t \in (0, T),\ x \in \partial\Omega, \\ &u(0) = u_0(x) && x \in \Omega. \end{aligned} \tag{3.1}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. Set $l(u(t, \cdot)) = \int_\Omega w(x)u(t, x)\,dx$, for some $w \in L^2(\Omega)$. To the diffusion operator Δ we can associate the sesquilinear form

$$\sigma(t)(\phi, \psi) = \int_\Omega \nabla\phi \cdot \nabla\psi\,dx,$$

for all $\phi, \psi \in V$. One can easily verify that (A1)–(A3) are satisfied. Suppose that F and g satisfy (A4) and (A5), respectively. Then (3.1) has a unique solution u on $[0, T]$. The following are examples of such functions: $F(t, u) = \gamma(t)(u^2/(1 + u^2))$, with $\gamma \in C[0, T]$, and $g(\xi) = (\delta_1 + \xi^2)/(\delta_2 + \xi^2)$, with $\delta_1, \delta_2 > 0$.

On the other hand, if F and g satisfy ($\overline{\text{A4}}$) and ($\overline{\text{A5}}$), then there exists a $0 < T^* \leq T$ such that (3.1) has a unique solution u on $[0, T^*]$. Examples of the functions F and g that satisfy ($\overline{\text{A4}}$) and ($\overline{\text{A5}}$) are $F(t, u) = \gamma(t)u^p$, $p \geq 1$, and $g(\xi) = 1/\xi^q$ on $[\alpha, \beta]$, where $\alpha, \beta, q > 0$.

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